A Geometrical Vector Algebra

BY T. PROCTOR HALL

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By

T. PROCTOR HALL, VANCOUVER, CANADA

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1. The laws of operation of any algebra are ultimately based upon its definitions. If the definitions are geometrical the algebraic operations have geometric correspondences. The operations of addition and subtraction in common algebra, for example, correspond to the geometric addition and subtraction of straight lines, vectors, surfaces, etc.

In this algebra new definitions of vector multiplication and division are adopted, in consequence of which all algebraic operations upon vectors (directed unlocated straight lines or steps), or rather upon vector symbols, correspond to geometric operations in space upon the vectors themselves; and every algebraic vector expression corresponds to some geometric configuration of the vectors themselves.

In every vector demonstration or problem, therefore, the student may think in terms of either algebra or geometry or both; and may at any time change from one realm of thot to the other with no break in the continuity.

This algebra is developed first in terms of analytical geometry for three-fold space, and is then adapted to two-fold and to four-fold space. Complex numbers, spherical trigonometry, and quaternion rotations, appear as special cases.

2. NOTATION.—Taking three rectangular axes X, Y, Z, let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ denote unit vectors (steps) outward from the centre O, along the axes. Unit vectors in the opposite direction from O are denoted by $\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}$. Vectors in general are herein denoted by black faced Gothic capitals, and the corresponding unit vectors by black faced italics. For purposes of designation and operation all vectors (unless otherwise indicated) are understood to start from O, the centre of coordinates.

Then if $\mathbf{A}$ is any vector, a is its length, $\boldsymbol{a}$ is unit length of the same vector, $a_x \boldsymbol{x}$, $a_y \boldsymbol{y}$, $a_z \boldsymbol{z}$ are the vector components of $\mathbf{A}$ along X, Y, Z, and a_x, a_y, a_z are the lengths of these components.

Then $\mathbf{A} = a\boldsymbol{a}$

$$= a_x \boldsymbol{x} + a_y \boldsymbol{y} + a_z \boldsymbol{z} \text{ by vector addition.}$$

$$a^2 = a_x^2 + a_y^2 + a_z^2 \text{ by solid geometry.}$$

The symbol $\mathbf{A}$ is used to indicate (1) the vector from O to the point whose rectangular coordinates are a_x, a_y, a_z; (2) motion from O to the extremity of $\mathbf{A}$; (3) a rotor, defined in §7.

The line or locus of $\mathbf{A}$ is expressed by an elongated l, thus $l\mathbf{A}$, and any part of this locus, from m to n, is written ${}_m l\,\mathbf{A}$. Surface loci are ordinarily expressed by two l's and solid loci by three l's.

3. To express the cosine of the angle between two vectors in terms of the coordinates of the vectors.

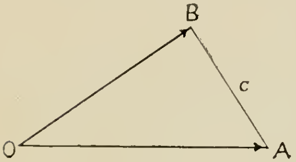


FIG. 1.

Let c be the length of the line joining the extremities of the vectors $\mathbf{A}$, $\mathbf{B}$, from O.

By solid geometry—

$$c^2 = (a_x - b_x)^2 + (a_y - b_y)^2 + (a_z - b_z)^2$$

By plane trigonometry—

$$c^2 = a^2 + b^2 - 2ab \cos \mathbf{A}\,\mathbf{B}.$$

Therefore, $\cos \mathbf{A}\,\mathbf{B} = \dfrac{a_x\, b_x + a_y\, b_y + a_z\, b_z}{a\, b} \equiv \dfrac{S_{ab}}{a\, b}$,

where $S_{ab} \equiv$ the sum of the $a\,b$ products $= a\,b \cos \mathbf{A}\,\mathbf{B}$.

If $S_{ab} = 0$, $\mathbf{A} \perp \mathbf{B}$, and conversely.

EXAMPLE 1.—Find the angle between the vectors $\mathbf{x} + 2\mathbf{y}$ and $2\mathbf{x} - \mathbf{y} + c\mathbf{z}$. Here $S = 0$, and the vectors are perpendicular.

EXAMPLE 2.—What angles does the vector $2\mathbf{x} - \mathbf{y} + \mathbf{z}$ $(= \mathbf{A})$ make with the axes $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$?

$$a = \sqrt{6};\ \therefore \cos \mathbf{A}\,\mathbf{x} = \frac{2}{\sqrt{6}},\quad \cos \mathbf{A}\,\mathbf{y} = \frac{-1}{\sqrt{6}},\quad \cos \mathbf{A}\,\mathbf{z} = \frac{1}{\sqrt{6}}.$$

4. ADDITION AND SUBTRACTION.—Addition is geometrically defined as the process of making the second vector step from the extremity of the first. The sum is the new vector from O to the extremity of the second vector thus added.

Algebraically addition is performed by resolving the vectors into their components and adding these.

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (a_x\,\mathbf{x} + a_y\,\mathbf{y} + a_z\,\mathbf{z}) + (b_x\,\mathbf{x} + b_y\,\mathbf{y} + b_z\,\mathbf{z}) \\ &= (a_x + b_x)\,\mathbf{x} + (a_y + b_y)\,\mathbf{y} + (a_z + b_z)\,\mathbf{z}. \end{aligned}$$

Subtraction is addition of the negative of a vector.

Hence, both geometrically and algebraically vector terms are commutative.

$$\mathbf{A} + \mathbf{B} = + \mathbf{B} + \mathbf{A}.$$

5. COLLINEAR VECTORS.—Two vectors **A**, **B**, are in the same line when

$$\mathbf{A} \equiv n\mathbf{B},$$

or

$$\frac{a_x}{b_x} = \frac{a_y}{b_y} = \frac{a_z}{b_z} = n.$$

If n is positive, **A** and **B** are in the same direction; if negative, **A** and **B** are opposite. If $n = 1$, $\mathbf{A} \equiv \mathbf{B}$.

6. COPLANAR VECTORS.—Three vectors, **A**, **B**, **C**, are in the same plane when another vector, **K**, can be found which is perpendicular to each of them. Then $S_{ak} = S_{bk} = S_{ck} = 0$, by §3. Eliminating k_x, k_y, k_z we get the coplanar equation

$$| a_x \; b_y \; c_z | = 0.$$

The determinant $| a_x \; b_y \; c_z |$ is six times the volume of the tetrahedron whose corners are O A B C. When this volume is zero **A**, **B**, **C**, are coplanar.

EXAMPLE.—Find the conditions under which **A** is perpendicular to $\mathbf{C} = \boldsymbol{x} + \boldsymbol{z}$, and in the **B C** plane where $\mathbf{B} = 2\boldsymbol{x} - \boldsymbol{y}\sqrt{3}$.

The condition of perpendicularity is

$$\mathbf{S}_{ac} = 0, \qquad \text{or } a_x + a_z = 0.$$

The coplanar equation is

$$\begin{vmatrix} 2 & -\sqrt{3} & 0 \\ 1 & 0 & 1 \\ a_x & a_y & a_z \end{vmatrix} = 0.$$

Therefore $\mathbf{A} = a_x\,(\boldsymbol{x} - \boldsymbol{y}\sqrt{3} - \boldsymbol{z})$.

7. MULTIPLICATION of the vector **B** by the vector **A** is written **A B**, and is defined geometrically as the combined operations,

(1) Extension of **B** until its length is ab,

(2) Simultaneous rotation of **B** thru 90° about **A** as an axis, in a direction which is right handed or clockwise when facing in the positive direction of **A**.

Each vector multiplier is a tensor-rotor. The rotor power of all vectors is the same and needs no separate expression at this stage.

The product **AB** is that vector from O whose extremity is the final position of the point **B** after extension and rotation. The locus of **AB** is the curve traced by the point **B** during the operation.

ABC means the operation of **A** on the product **BC**, or

$$\mathbf{ABC} \equiv \mathbf{A}.\ \mathbf{BC}.$$

Also $$\mathbf{A}^2\,\mathbf{B} \equiv \mathbf{A}.\ \mathbf{AB}, \text{ etc.}$$

It next becomes necessary to find the laws of algebraic multiplication that correspond to the geometric changes here defined.

8. Multiplication by a collinear vector makes no change except in length or sign,

$$\mathbf{x}\mathbf{x} = \mathbf{x}$$
$$\mathbf{x}\overline{\mathbf{x}} = \overline{\mathbf{x}}$$
$$\overline{\mathbf{x}}\mathbf{x} = \overline{\mathbf{x}}$$
$$\overline{\mathbf{x}}\,\overline{\mathbf{x}} = \mathbf{x}$$
$$\mathsf{A}\boldsymbol{a} = \mathsf{A}, \text{ etc.}$$

9. Unit perpendicular vectors give the following results which are geometrically evident.

$$\mathbf{x}\mathbf{y} = \mathbf{z} \qquad \overline{\mathbf{x}}\mathbf{y} = \overline{\mathbf{z}}$$
$$\mathbf{x}\mathbf{z} = \overline{\mathbf{y}} \qquad \overline{\mathbf{x}}\,\overline{\mathbf{z}} = \overline{\mathbf{y}}$$
$$\mathbf{x}\overline{\mathbf{y}} = \overline{\mathbf{z}} \qquad \overline{\mathbf{x}}\,\overline{\mathbf{y}} = \mathbf{z}$$
$$\mathbf{x}\overline{\mathbf{z}} = \mathbf{y} \qquad \overline{\mathbf{x}}\mathbf{z} = \mathbf{y}$$

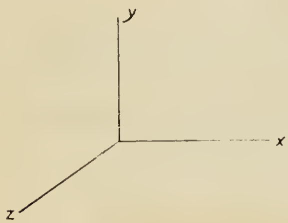


FIG. 2

and similarly for $\mathbf{y}$ and $\mathbf{z}$ as operators. Here the laws of signs are the same as in common algebra, so long as the factors are in alphabetical circular order;

$$\overline{\mathbf{x}}\mathbf{y} = \overline{\mathbf{z}} = \mathbf{x}\overline{\mathbf{y}},$$
$$\mathbf{x}\overline{\mathbf{y}} = \overline{\mathbf{z}} = \overline{\mathbf{x}}\mathbf{y} = -\,\mathbf{x}\mathbf{y}.$$

But reversing the order of the factors changes the sign of the product;

$$\mathbf{x}\mathbf{y} = \mathbf{z}$$
$$\mathbf{y}\mathbf{x} = \overline{\mathbf{z}}.$$

The second power of an unit perpendicular operator is equivalent to –1,

$$\mathbf{x}^2\mathbf{y} = \mathbf{x}\mathbf{z} = \overline{\mathbf{y}}$$
$$\overline{\mathbf{x}}^2\mathbf{y} = \overline{\mathbf{x}}\,\overline{\mathbf{z}} = \mathbf{y}.$$

The fourth power leaves the operand unchanged,

$$\mathbf{x}^4\mathbf{y} = \mathbf{x}^2\overline{\mathbf{y}} = \mathbf{y}.$$

When the vectors are not units the product of their tensors is prefixed to the vector product,

$$a\mathbf{x}.\ b\mathbf{y} = ab.\ \mathbf{x}\mathbf{y} = ab\mathbf{z}.$$

10. To find the algebraic product of any two vectors. Let the product be $\mathsf{K} = \mathsf{AB}$. Draw $KD \perp OA$, and OV equal and parallel to DK. Then the length OD is

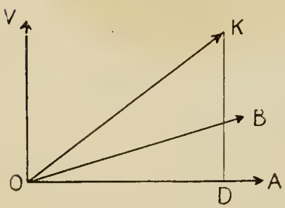


FIG. 3

$$OD = OK \cos \mathsf{AK}$$
$$= ab \cos \mathsf{AB}$$
$$= S_{ab}, \qquad \textit{by } \S 3.$$

As vectors $OK = OD + DK$,

or $\mathsf{K} = S_{ab}\,\boldsymbol{a} + \mathsf{V}$.

To determine V we have the equations of perpendicularity,

$$S_{av} = a_x v_x + a_y v_y + a_z v_z = o$$

$$S_{bv} = b_x v_x + b_y v_y + b_z v_z = o,$$

and from the triangle ODK,

$$v_x^2 + v_y^2 + v_z^2 = v^2 = a^2 b^2 - S^2.$$

Solving we get

$$v_x = a_y b_z - a_z b_y$$

$$v_y = a_z b_x - a_x b_z$$

$$v_z = a_x b_y - a_y b_x.$$

Hence the "Vector Normal" to A, B, is

$$\mathsf{V} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{vmatrix}$$

and its length is

$$v = \sqrt{a^2 b^2 - S^2} = a b \,\mathit{sin}\, \mathsf{AB}.$$

The product K is thus expressed in terms of the given vectors and their components, in the equation

$$\mathsf{AB} = S\boldsymbol{a} + \mathsf{V}.$$

Example. Find the product AB when

$$\mathsf{A} = 3\boldsymbol{y} - \boldsymbol{z}, \quad \mathsf{B} = 3\boldsymbol{z} - \boldsymbol{y}.$$

Here $\qquad S = -6, \quad \mathsf{V} = 8\boldsymbol{x}, \quad a = \sqrt{10},$

$$\therefore \mathsf{AB} = 8\boldsymbol{x} - \frac{3\sqrt{10}}{5}(3\boldsymbol{y} - \boldsymbol{z}).$$

11. PERMUTATION OF FACTORS. It is geometrically and algebraically evident that

$$S_{ab} = S_{ba}$$

and that $\qquad \mathsf{V}_{ab} = - \mathsf{V}_{ba}.$

Hence $\qquad \mathsf{BA} = S_{ba}\,\boldsymbol{b} + \mathsf{V}_{ba}$

$$= S_{ab}\,\boldsymbol{b} - \mathsf{V}_{ab}$$

which is not equal to AB.

Changing the order of the factors changes the vector product. Vectors are not permutable.

12. OPERAND DISTRIBUTIVE. To find the product $\mathsf{A}\,(\mathsf{B} \pm \mathsf{C})$

let $\qquad \mathsf{B} \pm \mathsf{C} = \mathsf{D},$

so that $\qquad d_x = b_x \pm c_x$

$$d_y = b_y \pm c_y$$

$$d_z = b_z \pm c_z.$$

Then
$$\begin{aligned}
\mathsf{A}\,(\mathsf{B} \pm \mathsf{C}) &= \mathsf{A}\mathsf{D}\\
&= S_{ad}\,\boldsymbol{a} + V_{ad}\\
&= (S_{ab} \pm S_{ac})\,\boldsymbol{a} + V_{ab} \pm V_{ac}\\
&= (S_{ab}\,\boldsymbol{a} + V_{ab}) \pm (S_{ac}\,\boldsymbol{a} + V_{ac})\\
&= \mathsf{AB} \pm \mathsf{AC}.
\end{aligned}$$

The operand is therefore distributive.

13. OPERATOR NOT DISTRIBUTIVE. To find the product $(\mathsf{A} \pm \mathsf{B})\mathsf{C}$, let $\mathsf{A} \pm \mathsf{B} = \mathsf{K}$ so that

$$k_x = a_x \pm b_x$$
$$k_y = a_y \pm b_y$$
$$k_z = a_z \pm b_z$$

and $$k^2 = a^2 + b^2 \pm 2\,S_{ab}, \text{ by § 3.}$$

Then
$$\begin{aligned}
(\mathsf{A} \pm \mathsf{B})\mathsf{C} &= \mathsf{K}\mathsf{C}\\
&= S_{kc}\,\boldsymbol{k} + V_{kc}\\
&= \frac{S_{ac} \pm S_{bc}}{\sqrt{a^2 + b^2 \pm 2\,S_{ab}}}\,(\mathsf{A} \pm \mathsf{B}) + (V_{ac} \pm V_{bc})
\end{aligned}$$

which is not equal to $\mathsf{AC} \pm \mathsf{BC}$.

Hence the operator is not in general distributive.

14. FACTORS MUST NOT CHANGE ASSOCIATION.

$$\begin{aligned}
\mathsf{ABC} &= \mathsf{A}\,(S_{bc}\,\boldsymbol{b} + V_{bc})\\
&= \frac{S_{bc}}{b}\,\mathsf{AB} + \mathsf{A}V_{bc}\\
&= \frac{S_{ab}\,S_{bc}}{a\,b}\,\mathsf{A} + \frac{S_{bc}}{b}\,V_{ab} + |\,a_x\,b_y\,c_z\,|\,\boldsymbol{a} + S_{ac}\,\mathsf{B} - S_{ab}\,\mathsf{C}.
\end{aligned}$$

To expand $\mathsf{AB.\,C}$, let $\mathsf{K} = \mathsf{AB}$, so that

$$\begin{aligned}
\mathsf{AB.\,C} = \mathsf{KC} &= S_{kc}\,\boldsymbol{k} + V_{kc}\\
&= \left\{\frac{S_{ab}\,S_{ac}}{a} + |\,a_x\,b_y\,c_z\,|\right\}\boldsymbol{a}\,\boldsymbol{b} + \frac{S_{ab}}{a}\,V_{ac}\\
&\quad + \begin{vmatrix} |\,a_y\,b_z\,|, & |\,a_z\,b_x\,|, & |\,a_x\,b_y\,| \\ c_x & c_y & c_z \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{vmatrix}
\end{aligned}$$

which is not equal to ABC.

Hence the association of a factor must not in general be altered. But if $\mathsf{C} = \mathsf{A}$ these two products become identical, and therefore

$$\mathsf{A.\,BA} = \mathsf{AB.\,A}.$$

15. POWERS OF AN OPERATOR.

$$\mathsf{A}\mathsf{B} = S\boldsymbol{a} + \mathsf{V}$$

$$\mathsf{A}^2\mathsf{B} = \mathsf{A}.\ \mathsf{A}\mathsf{B} = \mathsf{A}\ (S\boldsymbol{a} + \mathsf{V})$$

$$= S\mathsf{A} + \mathsf{A}\mathsf{V}$$

$$= 2\ S\mathsf{A} - a^2\mathsf{B} \text{ by expansion and multiplication.}$$

$$\mathsf{A}^3\mathsf{B} = \mathsf{A}\ (2\ S\mathsf{A} - a^2\mathsf{B})$$

$$= 2\ a^2\ S\boldsymbol{a} - a^2\ (S\boldsymbol{a} + \mathsf{V})$$

$$= a^2\ (S\boldsymbol{a} - \mathsf{V}), \text{ which is geometrically evident.}$$

$$\mathsf{A}^4\mathsf{B} = a^3\ S\boldsymbol{a} - a^2\ (S\boldsymbol{a} - a^2\mathsf{B})$$

$$= a^4\ \mathsf{B}, \text{ which is also geometrically evident.}$$

From these results it is easy to write the expansion of any value of $\mathsf{A}^n\mathsf{B}$ when n is a positive integer.

16. LAWS OF MULTIPLICATION, summary.

(1) Factors are not permutable (§11)

$\mathsf{A}\mathsf{B}$ is not equal to $\mathsf{B}\mathsf{A}$.

(2) The operand is distributive (§12)

$$\mathsf{A}\ (\mathsf{B} \pm \mathsf{C}) = \mathsf{A}\mathsf{B} \pm \mathsf{A}\mathsf{C},$$

but the operator is not (§13)

$(\mathsf{A} \pm \mathsf{B})\ \mathsf{C}$ is not equal to $\mathsf{A}\mathsf{C} \pm \mathsf{A}\mathsf{C}$.

(3) The association of a factor must not be changed (§14).

$\mathsf{A}.\ \mathsf{B}\mathsf{C}$ is not equal to $\mathsf{A}\mathsf{B}.\ \mathsf{C}$

but $\mathsf{A}.\ \mathsf{B}\mathsf{A} = \mathsf{A}\mathsf{B}.\mathsf{A}$.

(4) The fourth power of an operator is equivalent to the fourth power of its tensor (§15).

(5) The common laws of signs are true for opera ' and product; not for the operator.

17. PERPENDICULAR VECTORS. When $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are perpendicular, $S_{ab} = S_{ac} = S_{bc} = o$, (§3), and $\mathsf{A}\mathsf{B} = \mathsf{V}$, (§15).

Then the laws of §16 become the following:

(1) Permuting the factors, i.e., interchanging operator and operand, changes the sign of the product.

$$\mathsf{B}\mathsf{A} = \mathsf{V}_{ba} = -\mathsf{V}_{ab} = -\mathsf{A}\mathsf{B}.$$

(2) Both operator and operand are distributive.

$$(\mathsf{A} \pm \mathsf{B})\mathsf{C} = \mathsf{A}\mathsf{C} \pm \mathsf{B}\mathsf{C}$$

$$\mathsf{A}\ (\mathsf{B} \pm \mathsf{C}) = \mathsf{A}\mathsf{B} \pm \mathsf{A}\mathsf{C}.$$

(3) The association of a factor must not be changed.

$$\mathbf{A}.\,\mathbf{BC} \text{ is not equal to } \mathbf{AB}.\mathbf{C}.$$

$$\text{but } \mathbf{A}.\,\mathbf{BA} = \mathbf{AB}.\,\mathbf{A}.$$

(4) The square of an operator is –1 times the square of its tensor.

$$\mathbf{A}^2\mathbf{B} = -\,a^2\mathbf{B}.$$

(5) The common laws of signs hold true.

If $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ be any three unit perpendicular vectors in the same circular order as $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$; then $\boldsymbol{ab} = \boldsymbol{c}$, $\boldsymbol{bc} = \boldsymbol{a}$, $\boldsymbol{ca} = \boldsymbol{b}$, and these vectors may serve as units of the system, as well as $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$.

18. DIVISION is the inverse of multiplication, so that if

$$\mathbf{AB} = \mathbf{C}$$

$$\frac{\mathbf{C}}{\mathbf{A}} = \mathbf{A}^{-1}\mathbf{C} = \mathbf{B}.$$

Geometrically, division is a negative turn of 90° about the divisor (identical in this respect with multiplication by the negative of the divisor) and reduction in length to that given by the quotient of the tensors.

19. GENERAL FORMULA for $\mathbf{A}^n\mathbf{B}$ where n is real.

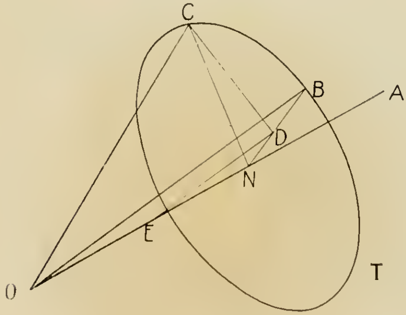


FIG. 4

Let O A, O B, be the vectors **A**, **B**, and let O C be in line with their vector product, so that $\mathbf{A}^n\mathbf{B} = a^n$ times O C.

Let B C T be the circle of revolution of **B** about **A**; N its centre; N B, N C, its radii. Draw C D $\perp$ N B; D E $\parallel$ B O. Let $\angle$ C N B $= \theta = n\frac{\pi}{2}$ be the angle of rotation of **B**.

Then $NC = NB = \frac{v}{a}$, where v is the length of $\mathbf{V}_{ab}$.

$$DC = \frac{v}{a}\ \sin\ \theta,$$

$$DE = BO\ \frac{ND}{NB} = b\ \cos\ \theta,$$

$$OE = DB\ \frac{ON}{NB} = \frac{v}{a}\ vers\ \theta.\ cot\ \mathbf{AB}$$

$$= \frac{v}{a}\ vers\ \theta.\ \frac{s}{v} = \frac{s}{a}\ vers\ \theta.$$

As vectors $\quad \mathrm{OC} = \mathrm{OE} + \mathrm{ED} + \mathrm{DC}$

$$= \frac{\mathrm{S}}{a}\,\boldsymbol{a}\ vers\,\theta + \mathsf{B}\ cos\ \theta + \frac{\mathsf{V}}{a}\ sin\ \theta$$

$$\therefore\ \mathsf{A}^n\mathsf{B} = a^n.\ \mathrm{OC}$$

$$= a^{n-1}\,(\mathrm{S}\boldsymbol{a}\ vers\,\theta + a\,\mathsf{B}\ cos\ \theta + \mathsf{V}\ \ sin\ \theta).$$

This formula, being true for all real values of n, includes products, quotients, powers and roots of vector operators.

EXAMPLE.—Two rods, A and B, are joined at one end. A is one foot long, and the perpendicular distance of its free end from B is six inches. B is turned 60° about the axis of A, then A is turned 90° in the same direction about the new axis of B. Find the new position of A.

Let the joined ends be at O. Let $\mathsf{B} = b\boldsymbol{x}$, and $\mathsf{A} = a_x\,\boldsymbol{x} + a_y\,\boldsymbol{y}$. Since $a = 1$, and $a_y = \frac{1}{2}$, $\quad \mathsf{A} = \frac{1}{2}\,(\boldsymbol{x}\sqrt{3} + \boldsymbol{y})$.

The result of the first rotation is represented by

$$\mathsf{C} = \mathsf{A}^{\frac{2}{3}}\,\mathsf{B} = \mathrm{S}\boldsymbol{a}\ vers\ 60° + \mathsf{B}\ cos\ 60° + \mathsf{V}\ sin\ 60°$$

$$= \frac{b}{8}\,(7\boldsymbol{x} + \boldsymbol{y}\sqrt{3} - 2\boldsymbol{z}\sqrt{3}).$$

The second rotation is

$$\boldsymbol{c}\,\mathsf{A} = \mathrm{S}\,\boldsymbol{c} + \mathsf{V}_{ca}$$

$$= \frac{1}{16}\,(9\,\boldsymbol{x}\sqrt{3} - 3\boldsymbol{y} - 2\,\boldsymbol{z}),$$

which gives the final position of the free end of A.

20. QUATERNIONS.

When $\quad \mathsf{A} \perp \mathsf{B},\ \mathrm{S} = 0$

and $\qquad \mathsf{V} = \mathsf{A}\mathsf{B}. \qquad (\S 17).$

Then $\qquad \mathsf{A}^n\mathsf{B} = a^n\ (\mathsf{B}\ cos\ \theta + \boldsymbol{a}\,\mathsf{B}\ sin\ \theta)$

$$= a^n\ (cos\ \theta + \boldsymbol{a}\ sin\ \theta)\ \mathsf{B}.$$

Now $\boldsymbol{a}^2$ as a perpendicular operator is equivalent to -1; and by expansion in series, exactly as with the complex $(cos\ \theta + i\ sin\ \theta)$, it may be shown that the rotor of A^n

$$cos\ \theta + \boldsymbol{a}\ sin\ \theta = e^{\boldsymbol{a}\theta}$$

where θ is the angle and $\boldsymbol{a}$ the axis of rotation.

Hence for perpendicular vectors

$$\mathsf{A}^n\mathsf{B} = a^n\,e^{\boldsymbol{a}\theta}\ \mathsf{B}.$$

The operator A^n is a tensor-rotor-vector, or a directed quaternion, when applied to vectors perpendicular to A. It has the four fundamental characters of a quaternion, namely,

(1) Since $\mathsf{A}^n\ \mathsf{B} = \mathsf{C}$, A^n may be regarded as the ratio of C to B;

(2) It is the product of a tensor and a directed rotor, $a^n.\ e^{\boldsymbol{a}\theta}$;

(3) It is the sum of a scalar or number and a directed unlocated line or vector, $a^n \cos\theta + a^n\, \boldsymbol{a} \sin\theta$;

(4) It is a quadrinomial of the form $k + l\boldsymbol{x} + m\boldsymbol{y} + n\boldsymbol{z}$, where k is a pure number and the directive units $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, have the relations

$\boldsymbol{x}^2 = \boldsymbol{y}^2 = \boldsymbol{z}^2 = \boldsymbol{xyz} = -1.$

21. VECTOR ARCS. The rotor $e^{\boldsymbol{a}}$ turns thru the angle a about the axis A any vector in the plane perpendicular to A. The index $\boldsymbol{a}$ is a vector angle whose axis is A and whose magnitude is a radians. The length of the subtended arc is $a\,a$. If this circular arc be taken as a vector, written $\boldsymbol{a}$, it is understood that its angle is a, its axis A and its radius a. A vector arc may take any position in its own circle, and has therefore one more degree of freedom than its vector axis.

Vector arcs need not be confined to arcs of circles, but whether the extension to other curves would be of any particular value remains to be seen. A rough classification gives the following:

(1) Straight vectors,

(2) Plane vectors, having single curvature.

A. Conic,

a. Circular, *b.* Elliptic, *c.* Parabolic, *d.* Hyperbolic,

B. Spiral, etc.

(3) Solid vectors, with double curvature.

22. SUM OF CIRCULAR AND STRAIGHT VECTORS.

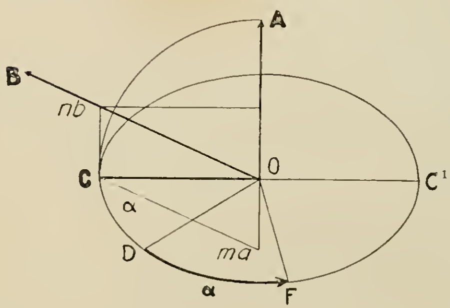


Fig. 5.

Let the plane of the arc $\boldsymbol{a}$ meet the plane of A, B, in the line $\mathsf{C}\,\mathsf{C}^1$; let C be so chosen that $\angle\ \mathsf{B}\mathsf{C}$ is not greater than $90°$, i. e., so that n is positive; and let $a = c$.

Let $\quad \mathbf{C} = m\mathbf{A} + n\mathbf{B}.$

Then from the figure

$$n^2 b^2 - m^2 a^2 = c^2 = a^2,$$

$$\cos \mathbf{AB} = \frac{ma}{nb} = \frac{S_{ab}}{ab} \qquad (\S 3)$$

$$\therefore n = \frac{a^2}{v}, \quad m = \frac{S}{v},$$

$$\therefore \mathbf{C} = \frac{1}{v}(S\,\mathbf{A} + a^2\mathbf{B}).$$

Let $\qquad \angle\, COD = a^1.$

Then $\qquad \boldsymbol{a} = \mathbf{F} - \mathbf{D} = e^{\boldsymbol{a}^1+\boldsymbol{a}}\mathbf{C} - e^{\boldsymbol{a}^1}\mathbf{C} = (e^{\boldsymbol{a}^1+\boldsymbol{a}} - e^{\boldsymbol{a}^1})\,\mathbf{C}$

$$= \left\{\cos(a^1 + a) + \boldsymbol{a}\sin(a^1 + a) - \cos a^1 - \boldsymbol{a}\sin a^1\right\}\mathbf{C}$$

$$= 2\sin\frac{a}{2}\left\{\sin\left(a^1 + \frac{a}{2}\right) + \boldsymbol{a}\cos\left(a^1 + \frac{a}{2}\right)\right\}\mathbf{C}.$$

To this $\mathbf{B}$ is readily added.

If $\mathbf{B}$ is parallel to $\mathbf{A}$, $\mathbf{C}$ is indeterminate and any radius of the $\boldsymbol{a}$ circle may be taken as $\mathbf{C}$. In this case the sum is a point on a right helix or screw whose axis is $\mathbf{A}$. Since the addition may begin at any point of the $\boldsymbol{a}$ circle, the sum is a screw vector whose radius, pitch and direction are fixed.

23. SUM OF TWO CIRCULAR VECTORS. Let $\boldsymbol{a}$, $\boldsymbol{\beta}$ be two circular vectors with a common centre O; and let $\mathbf{C} = V_{ab}$ be the intersection of their planes. Let $\quad CB_o = \beta^1, \quad CA_o = a^1.$

Then $\qquad \boldsymbol{a} = \mathbf{A}_1 - \mathbf{A}_o = (e^{\boldsymbol{a}^1+\boldsymbol{a}} - e^{\boldsymbol{a}^1})\,a\,\mathbf{c},$

$$\boldsymbol{\beta} = \mathbf{B}_1 - \mathbf{B}_o = (e^{\boldsymbol{\beta}^1+\boldsymbol{\beta}} - e^{\boldsymbol{\beta}^1})\,b\mathbf{c}.$$

Any third circular vector whose position is determined with reference to the intersection of its plane with the plane of $\boldsymbol{a}$ or $\boldsymbol{\beta}$, may be similarly expressed and the sum readily found. In expanding these expressions it is convenient to remember that

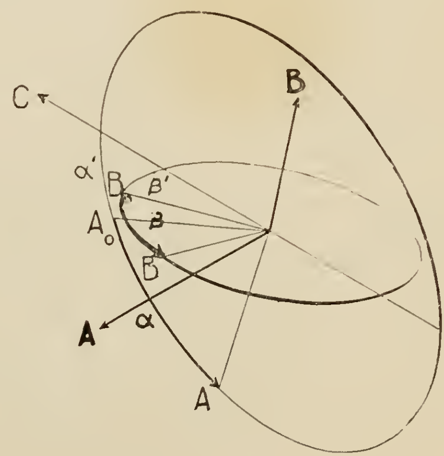


FIG. 6

when $\quad \mathsf{C} = \mathsf{V}_{ab}$

then $\quad \mathsf{AC} = \mathsf{V}_{ac} = \mathrm{S}_{ab}\,\mathsf{A} - a^2\,\mathsf{B}$

and $\quad \mathsf{BC} = \mathsf{V}_{bc} = b^2\,\mathsf{A} - \mathrm{S}_{ab}\,\mathsf{B}.$

When $\quad \boldsymbol{\beta} \equiv \boldsymbol{a}$ the sum is

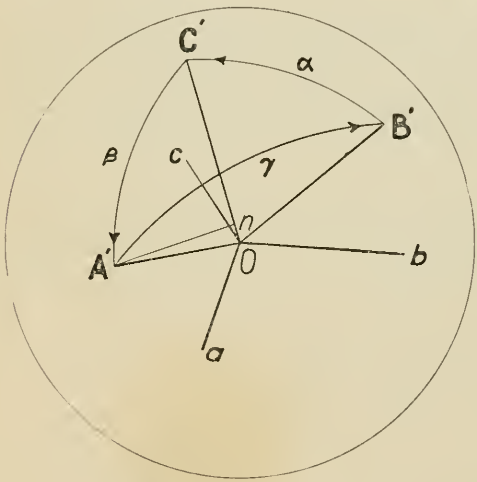
$$2\,\boldsymbol{a} = \boldsymbol{a}.\ 2\,a$$

which is a vector arc with angle a and radius $2a$.

The locus of the sum of two equal vector arcs beginning at the same point of intersection, when the planes are not identical, is an ellipse.

Also $\int (\boldsymbol{a} - \boldsymbol{a})$ is a straight line.

24. SPHERICAL TRIANGLE. Assume a sphere of unit radius, and upon it arcs of great circles. As an illustration of vector treatment let it be required to find the relation between the sines of the angles of a spherical triangle.



Let $\boldsymbol{a}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, be three circular vectors forming a spherical triangle; a, b c, their vector axes; A^1, B^1, C^1, the vectors from O to the angular points; A, B, C, the angles of the spherical triangle.

FIG. 7

Draw $A^1 n \perp O C^1$.

Then as vectors $O A^1 = O n + n A^1$

or $\quad \mathsf{A}^1 = \mathsf{C}^1 + \boldsymbol{\beta} = \mathsf{C}^1 \cos\beta + \boldsymbol{v}_{b^1c^1} \sin\beta.$

Similarly $\quad \mathsf{A}^1 = \mathsf{B}^1 - \boldsymbol{\gamma} = \mathsf{B}^1 \cos\gamma - \boldsymbol{v}_{c^1b^1} \sin\gamma.$

By inspection of the figure it is evident that in any spherical triangle

$$\boldsymbol{a} = \boldsymbol{v}_{b^1c^1}$$

$$\cos a = \mathrm{S}_{b^1c^1}, \qquad \sin a = v_{b^1c^1}$$

$$\cos \mathrm{A} = -\cos(\pi - \mathrm{A}) = -\mathrm{S}_{bc}, \quad \sin \mathrm{A} = v_{bc}.$$

$$\mathsf{A}^1 = \boldsymbol{v}_{bc} = \frac{|\, b_x\ c_y\ \boldsymbol{z}\,|}{v_{bc}}$$

$$= a^1_x\,\boldsymbol{x} + a^1_y\,\boldsymbol{y} + a^1_z\,\boldsymbol{z}.$$

Similar equations may be written for the corresponding elements of the triangle.

From the last equation, equating coefficients of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$,

$$a^1_x = \frac{|\, b_y \; c_z \,|}{v_{bc}}, \quad a^1_y = \frac{|\, b_z \; c_x \,|}{v_{bc}}, \quad a^1_z = \frac{|\, c_x \; b_y \,|}{v_{bc}}.$$

Similarly,

$$b^1_x = \frac{|\, c_y \; a_z \,|}{v_{ca}}, \quad b^1_y = \frac{|\, c_z \; a_x \,|}{v_{ca}}, \quad b^1_z = \frac{|\, c_x \; a_y \,|}{v_{ca}}.$$

Then $\quad \cos\gamma = S_{a^1b^1} = \dfrac{S_{bc}\, S_{ca} - S_{ab}}{v_{bc}\, v_{ac}}$

and $\quad \dfrac{\sin\gamma}{\sin C} = \dfrac{\sqrt{1 - S^2_{a^1b^1}}}{v_{ab}} = \dfrac{\sqrt{1 - S^2_{ab} - S^2_{bc} - S^2_{ca} + 2\, S_{ab}\, S_{bc}\, S_{ca}}}{v_{ab}\, v_{bc}\, v_{ca}}.$

The last expression is symmetrical in a, b, c, and therefore

$$\frac{\sin\alpha}{\sin A} = \frac{\sin\beta}{\sin B} = \frac{\sin\gamma}{\sin C}.$$

25. CONIC VECTORS are expressible in terms of the radius vector from the focus to each extremity of the segment of the curve.

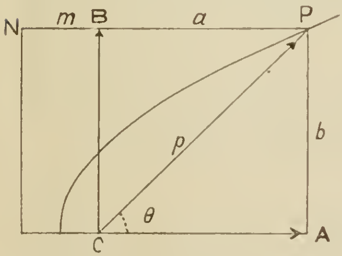


FIG. 8

Let **A** be the axis of a conic, O its focus, N its directrix, **P** the radius vector, a, b the coordinates of **P** with reference to **A** and **B**; and let $p = e\,(a + m)$, where $e = \dfrac{1}{c}$ is the eccentricity.

Then $\quad a = cp - m$

$$b^2 = p^2 - a^2 = p^2(1 - c^2) + 2cmp - m^2.$$

$\therefore \mathbf{P} = \mathbf{A} + \mathbf{B}$

$$= (cp - m)\,\boldsymbol{a} + \boldsymbol{b}\sqrt{[p^2(1 - c^2) + 2cmp - m^2]}.$$

The conic vector from P_o to P is $\mathbf{P} - \mathbf{P}_o$.

P may also be expressed in terms of a, b, or θ.

Thus $\quad a = p\cos\theta, \qquad cp = a + m$

$$b = p\sin\theta, \qquad = p\cos\theta + m,$$

$$p = \frac{m}{c - \cos\theta}.$$

$$\therefore \mathbf{P} = \frac{m}{c - \cos\theta}\,(\boldsymbol{a}\cos\theta + \boldsymbol{b}\sin\theta).$$

If p is a constant, $c = m = \infty$, then for the circle

$$\mathsf{P} = p\ (\boldsymbol{a}\ cos\ \theta + \boldsymbol{b}\ sin\ \theta).$$

Similarly in any conic

$$\mathsf{P} = \mathsf{A} + \frac{\boldsymbol{b}}{c}{}^{1}\left\{(a + m)^2 - a^2 c^2\right\}.$$

Also $$\mathsf{P} = \frac{m + c_{1}\ [m^2 + b^2 (c^2 - 1)]}{c^2 - 1}\ \boldsymbol{a} + \mathsf{B}.$$

but when $c = 1$, in the parabola,

$$\mathsf{P} = \frac{b^2 - m^2}{2m}\ \boldsymbol{a} + \mathsf{B}.$$

We have then an expression for any conic vector as the difference of two straight vectors, $\mathsf{P} = \mathsf{P}_o$; which may be expressed in terms of either of the variables, a, b or θ.

The sum of two or more conic vectors would express approximately for a short distance the course of a body moving under gravitational forces from two or more sources. Whether this method of calculating would be an improvement on present methods I am not prepared to say.

Multiplication of a straight vector by a vector arc involves double curvature, and the locus of such a product is a convenient form by which to express solid vectors (§21). Again the utility is problematical.

26. DIFFERENTIATION OF STRAIGHT VECTORS. Any vector, A, may vary in length and in direction. Its variation may be expressed in terms of a for length and $\boldsymbol{a}$ for direction; or it may be expressed in terms of the components A_x, A_y, A_z.

Since the infinitesimal increments of a vector are also vectors, it is evident that by vector addition

$$\begin{aligned} d\mathsf{A} &= d_x\,\mathsf{A} + d_y\,\mathsf{A} + d_z\,\mathsf{A} \\ &= \boldsymbol{x}\ da_x + \boldsymbol{y}\ da_y + \boldsymbol{z}\ da_z \qquad (1) \end{aligned}$$

since $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, are absolute constants.

Also, $$\begin{aligned} d\mathsf{A} &= d_{\boldsymbol{a}}\,\mathsf{A} + d_a\,\mathsf{A} \\ &= a\ d\boldsymbol{a} + \boldsymbol{a}\ da. \qquad (2) \end{aligned}$$

It follows from (1) that the differential of a vector is the sum of the differentials of its components, and hence that differentiation is distributive over vector terms.

It follows from (2) that the ordinary rule for differentiation of a product holds true for any unit vector and its tensor, and hence for any product of a tensor and a vector.

27. To find the differential coefficient of a vector product, $\mathsf{A}^n\mathsf{B}$.

$$\mathsf{A}^n\mathsf{B} = a^{n-1}\,(\mathrm{S}\boldsymbol{a}\ vers\ \theta + a\,\mathsf{B}\ cos\ \theta + \mathsf{V}\ sin\ \theta).$$

Differentiating both sides of the equation with respect to $\theta, = n\frac{\pi}{2}$,

$$\frac{d}{d\theta}\mathsf{A}^n\mathsf{B} = \frac{2}{\pi}\mathsf{A}^n\mathsf{B}.loga + a^{n-1}(\mathrm{S}\boldsymbol{a}\ sin\ \theta - a\,\mathsf{B}\ sin\ \theta + \mathsf{V}\ cos\ \theta)$$

$$= \frac{2}{\pi}\mathsf{A}^n\mathsf{B}\ loga + a^{n-1}\,(cos\ \theta + \boldsymbol{a}\ sin\ \theta)\ \mathsf{V},$$

since, by multiplication, $\mathrm{S}\boldsymbol{a} - a\,\mathsf{B} = \boldsymbol{a}\,\mathsf{V}$.

The last term of the differential coefficient may also be written $a^{n-1}\,e^{a\theta}\,\mathsf{V}$. It is a tensor and rotor product of V (the vector normal of A and B), whose rotation is about A. $e^{a\theta}\,\mathsf{V}$ expresses the rotation of V in the plane perpendicular to A. $a^{n-1}e^{a\theta}\,\mathsf{V}$ traces a spiral in this plane, and as a vector it gives at any point the direction and rate of motion in this plane made by the point B, supplementary to the increase in the length of B.

The term $\frac{2}{\pi}\mathsf{A}^n\mathsf{B}\,loga$ is a multiple of the vector product, and for any given value of n it expresses the rate and direction of the increase, in length only, of that product.

The sum of the two terms gives the rate and direction of the motion of the point B for unit increase in θ. It is the vector tangent to the curve traced by $\mathsf{A}^n\mathsf{B}$, namely, the curve $\angle\mathsf{A}^n\mathsf{B}$.

EXAMPLE.—Find the tangent where the flat spiral $(a\,\boldsymbol{x})^n.\,b\,\boldsymbol{y}$ cuts the Y axis. The tangent is

$$\mathsf{T} = \frac{d}{d\theta}(a\,\boldsymbol{x})^n\ b\,\boldsymbol{y},$$

$$= a^n b\ \left[\left(\frac{2}{\pi}\ log\,a\ \ cos\ \ \theta - sin\,\theta\right)\boldsymbol{y} + \left(\frac{2}{\pi}\ log\,a.\ sin\ \theta - cos\ \theta\right)\boldsymbol{z}\right]$$

At the starting point $\theta = o$, and

$$\mathsf{T}_o = b\ \left(\frac{2}{\pi}\ log\,a.\boldsymbol{y} + \boldsymbol{z}\right).$$

When $n = 2$, $\theta = \pi$, and

$$\mathsf{T}_2 = -\,a^2\,\mathsf{T}_o.$$

When $n = 4$, $\theta = 2\pi$, and

$$\mathsf{T}_4 = a^4\,\mathsf{T}_0 = -\,a^2\,\mathsf{T}_2 .$$

This vector tangent makes at all times a constant angle with its radius, and its length gives the velocity of the generating point when the angular velocity is unity.

28. CURVATURE. If l be the length of a curve, and T the vector tangent, the curvature K is $\dfrac{d\mathsf{T}}{dl}$, and the radius of curvature is

$$\mathsf{R} = \frac{-\,\mathsf{K}}{k^2} .$$

29. LINEAR LOCI are loci having only one degree of freedom; lines or discrete points. A few examples are given:

${}_{a_0}\ell^{a}\,\mathsf{A} + \mathsf{B}$ is any part of the straight line drawn from the point B in the direction A.

${}_{n=0}\ell^{n=4}\,\mathsf{A}^n\mathsf{B} + \mathsf{C}$, when $\mathsf{A} \perp \mathsf{B}$, is a circle with centre C, radius b, and plane perpendicular to $\boldsymbol{a}$.

${}_{0}\ell^{2\pi}\,(\mathsf{A}\,cos\,\theta + \mathsf{B}\,sin\,\theta) + \mathsf{C}$ is an ellipse parallel to the $\mathsf{A\,B}$ plane.

${}_{0}\ell^{n}\,(\mathsf{A}^n\mathsf{B} + n\,\mathsf{C})$ includes a variety of curves.

If $\mathsf{A} \perp \mathsf{B}$, $\mathsf{C} \parallel \mathsf{A}$, and $a = 1$, the locus is a helix.

If $\mathsf{C} \perp \mathsf{A}$ the locus varies from a circle (when $c = o$) to a straight line (when $c = \infty$), passing thru the cycloid. In other positions of C the helix is acute angled. When $a > 1$ the curves are expanding and when $a < 1$ diminishing.

30. EXAMPLES OF SURFACE LOCI.

${}_{0}\ell^{a}\,{}_{0}\ell^{b}\,(\mathsf{A} + \mathsf{B}) + \mathsf{C}$ is a parallelogram whose adjacent sides A, B, start at the point C. Its diagonals are $\mathsf{A} \pm \mathsf{B}$. Its area is $ab\;sin\;\mathsf{A\,B} = v$. (§10).

${}_0\mathit{l}^{2}\,{}_0\mathit{l}^{4}\,\boldsymbol{b}^n \boldsymbol{a}^m \mathsf{B}$ is a closed surface, spherical if $\boldsymbol{a} \perp \boldsymbol{b}$, with radius b.

${}_0\mathit{l}^{4}\,{}_0\mathit{l}^{b}\,\boldsymbol{a}^n \mathsf{B}$ is the conical surface traced by B as it is turned about A.

${}_0\mathit{l}^{a}\,{}_0\mathit{l}^{b}(\mathsf{A} + (1-a)\,\mathsf{B})$ is the triangle OAB.

31. EXAMPLES OF SOLID LOCI.

$\mathit{lll}\,(\mathsf{A} + \mathsf{B} + \mathsf{C}) + \mathsf{D}$ is any parallelopiped.

Its diagonals are $\mathsf{A} + \mathsf{B} + \mathsf{C}$, $\mathsf{A} + \mathsf{B} - \mathsf{C}$, $\mathsf{A} - \mathsf{B} + \mathsf{C}$, $-\mathsf{A} + \mathsf{B} + \mathsf{C}$. Its volume is $|\,a_x\, b_y\, c_z\,|$. If $\mathsf{P} = \mathsf{V}_{ab}$ and $\mathsf{Q} = \mathsf{V}_{ac}$, the dihedral angle, a, over the edge A is found from the equation $\mathrm{S}_{pq} = p\,q\,cos\,a$.

${}_0\mathit{l}^{2}\,{}_0\mathit{l}^{4}\,{}_{n_0}\mathit{l}^{a}\,\boldsymbol{a}^n \boldsymbol{b}^m \mathsf{A}$ is a shell, spherical if $\mathsf{A} \perp \mathsf{B}$.

${}_0\mathit{l}^{4}\,\boldsymbol{a}^n \left\{ {}_0\mathit{l}^{4}\,\boldsymbol{b}^m\, {}_{c_0}\mathit{l}^{c}\,\mathsf{C} + \mathsf{R} \right\}$ is a hollow annulus if $\mathsf{B} \perp \mathsf{C}$, $\mathsf{R} \perp \mathsf{C}$, $\mathsf{A} \perp \mathsf{R}$, $\mathsf{A} \perp \mathsf{B}$.

32. THE REGION COMMON to two loci is found by equating the coefficients of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, in the expressions for the loci. If these equations are consistent, giving real values for the variables, the limits thus found are inserted in either of the loci to give the required locus of intersection.

EXAMPLE 1.—Find the region common to the straight line

$${}_0\mathit{l}^{n}\, n\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{y}, \text{ and the curve}$$

$${}_0\mathit{l}^{a}\left\{ a\,\boldsymbol{x} + (2\boldsymbol{x} + \boldsymbol{y})\, sin\ a \right\}.$$

Equating coefficients,

$$n = a + 2\ sin\ a$$

$$\tfrac{1}{2} = sin\ a.$$

Whence $n = a + 1 = arcsin\ \tfrac{1}{2} + 1$.

Inserting these values, both loci become

$${}_0\mathit{l}^{a}(\tfrac{1}{2}\,\boldsymbol{y} + \boldsymbol{x}\ arcsin\ \tfrac{1}{2})$$

which is a row of discrete points parallel to x.

EXAMPLE 2.—Find what part of the helix

$$\ell_0^n (\mathbf{x}^n \mathbf{y} + 3n\,\mathbf{x}) \equiv \ell_0^n (3\,n\,\mathbf{x} + \mathbf{y}\ cos\ \frac{n\pi}{2} + \mathbf{z}\ sin\ \frac{n\pi}{2})$$

is within the figure

$$\ell_0^4 \ell_0^c \ell_0^b (\mathbf{y}^m c\,\mathbf{z} + b\,\mathbf{x} - \mathbf{y}) \equiv \ell_0^4 \ell_0^c \ell_0^b \left\{ \mathbf{x}\,(c\,sin\,\frac{m\pi}{2} + b) - \mathbf{y} + \mathbf{z}\,c\,cos\,\frac{m\pi}{2} \right\}.$$

Equating coefficients of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$,

(1) $3n = c\ sin\ \frac{m\pi}{2} + b$

(2) $cos\ \frac{n\pi}{2} = -1, \quad \therefore\ sin\ \frac{n\pi}{2} = 0$, and $n = 2,\ 6,\ 10 \ldots\ldots$

(3) $c\ cos\ \frac{m\pi}{2} = sin\ \frac{n\pi}{2} = 0,$

If $c = 0,\ 3n = b.$

If $cos\,\frac{m\pi}{2} = 0,\ sin\,\frac{m\pi}{2} = \pm 1$, and since c is positive $3n = b + c.$

Inserting these values in the locus of the helix we get for the intersection a row of points

$\ell\,3\,n\,\mathbf{x} - \mathbf{y}$, where n has the values

2, 6, 10……….up to $\frac{b+c}{3}$.

EXAMPLE 3.—Find the intersection of the plane

$$\ell\ell\,(m\,\mathbf{x} + n\,\mathbf{z}) + 3\,\mathbf{x}$$

with the solid

$$\ell_0^2 \ell_0^a \ell_0^b \mathbf{x}^n\,(a\,\mathbf{x} + b\,\mathbf{y}) \equiv \ell_0^\pi \ell_0^a \ell_0^b \left\{ a\,\mathbf{x} + b\,\mathbf{y}\ cos\,\theta + b\,\mathbf{z}\ sin\theta \right\}.$$

Equating coefficients of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$,

(1) $m + 3 = a, \quad$ or $m = a - 3.$

(2) $b\ cos\ \theta = 0, \quad \therefore\ b = 0$, or $sin\ \theta = \pm 1.$

(3) $n = b\ sin\ \theta, = 0\quad$ or $\pm\, b.$

$\therefore\quad n = \pm\, b.$

Substituting in the locus of the plane we get for the intersection the parallelogram

$$\ell_0^a \ell_0^b (a\,\mathbf{x} \pm b\,\mathbf{z}).$$

EXAMPLE 4.—Find the locus of the intersection of the cube

$$\int_0^1\int_0^1\int_0^1 (a_x \boldsymbol{x} + a_y \boldsymbol{y} + a_z \boldsymbol{z})$$

with a plane which cuts its diagonal

$\mathsf{A} = \boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}$ perpendicularly.

Let $$\mathsf{B} = \boldsymbol{x} - \boldsymbol{y}$$

be one vector in the perpendicular plane, and

$$\mathsf{C} = \mathsf{V}_{ab} = \boldsymbol{x} + \boldsymbol{y} - 2\boldsymbol{z}$$

the other. The plane is

$$\int\int (l\,\mathsf{B} + m\,\mathsf{C} + n\,\mathsf{A})$$

where n is an arbitrary constant expressing the fractional distance from O to the point where the diagonal is cut.

Equate coefficients of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, in the two loci,

$$a_x = l + m + n$$
$$a_y = -l + m + n$$
$$a_z = -2m + n.$$

Therefore $$l = \frac{a_x - a_y}{2}$$

$$m = \frac{a_x + a_y - 2a_z}{6}$$

$$3n = a_x + a_y + a_z.$$

If $n = 0$ the plane goes thru O. Since a_x, a_y, a_z, are all positive and the sum zero, each of them is zero, and the locus of intersection is the point O.

If $n = 1$ the point of intersection is A.

If $n = \frac{1}{3}$, so that $a_x + a_y + a_z = 1$, while each varies between 0 and 1 subject to this condition, the locus is an equilateral triangle whose corners are found by giving to a_x, a_y, a_z, separately the maximum value, 1, in the expanded expression for the plane

$$\int_0^1\int_0^1\int_0^1 \left\{\frac{a_x - a_y}{2}\mathsf{B} + \frac{a_x + a_y - 2a_z}{6}\mathsf{C}\right\} + \tfrac{1}{3}\mathsf{A}.$$

If $n = \frac{2}{3}$ the locus is a similar triangle.

If $n = \frac{1}{2}$ the locus a regular hexagon.

33. PROJECTIONS. To express any vector K in terms of three non-coplanar vectors $\mathsf{A}, \mathsf{B}, \mathsf{C}$, write

$$l\mathsf{A} + m\mathsf{B} + n\mathsf{C} = \mathsf{K}$$

$$\therefore \quad l\,a_x + m\,b_x + n\,c_x = k_x$$

$$l\,a_y + m\,b_y + n\,c_y = k_y$$

$$l\,a_z + m\,b_z + n\,c_z = k_z$$

$$\therefore \quad l = \frac{|\,k_x\,b_y\,c_z\,|}{|\,a_x\,b_y\,c_z\,|},\quad m = \frac{|\,k_x\,c_y\,a_z\,|}{|\,a_x\,b_y\,c_z\,|},\quad n = \frac{|\,k_x\,a_y\,b_z\,|}{|\,a_x\,b_y\,c_z\,|}.$$

If we now write $n = 0$, $l\mathsf{A} + m\mathsf{B}$ is the projection of K, made parallel to C, upon the plane of A, B.

If A and B only are given, and the projection is desired of K perpendicularly upon A, B, take $\mathsf{C} = \mathsf{V}_{ab} = |\,a_x\,b_y\,\boldsymbol{z}\,|$, and proceed as before.

To project K in the direction of C upon a plane perpendicular to C, take any vector A, $\perp$ C, so that $S_{ac} = 0$,

as, $\quad \mathsf{A} = c_y\boldsymbol{x} - c_x\,\boldsymbol{y}$

and a second vector B, $= \mathsf{V}_{ac}$:

$$\mathsf{B} = \begin{vmatrix} c_x & c_y & c_z \\ c_y & -c_x & 0 \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{vmatrix}.$$

Then express K in terms of $\mathsf{A}, \mathsf{B}, \mathsf{C}$, as before.

The most general form of a locus is $\mathit{lll}\,\mathsf{K} + \mathsf{M}$, which is projected in the same way.

EXAMPLE.—Project upon the Y Z plane and parallel to D the helix

$$l\mathsf{B} = {}_0l^n(\boldsymbol{x}^n b\,\boldsymbol{y} + a\,n\,\boldsymbol{x})$$

$$= {}_0l^n\{b\,(\boldsymbol{y}\cos\theta + \boldsymbol{z}\sin\theta) + a\,n\,\boldsymbol{x}\}.$$

Let $\mathsf{B} = l\boldsymbol{y} + m\boldsymbol{z} + r\mathsf{D}$.

Equating the coefficients of $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$,

$$r\,d_x = a\,n$$

$$l + r\,d_y = b\cos\theta$$

$$m + r\,d_z = b\sin\theta.$$

$$\therefore \quad \mathsf{B}^1 = \boldsymbol{y}\left\{b\cos\theta - a\,n\frac{d_y}{d_x}\right\} + \boldsymbol{z}\left\{b\sin\theta - a\,n\frac{d_z}{d_x}\right\}$$

the locus of which is the required projection.

34. PLANE ALGEBRA. Every vector in the X Y plane is of the form

$$\begin{aligned} \mathsf{A} &= a_x\,\boldsymbol{x} + a_y\,\boldsymbol{y} \\ &= a_x\,\boldsymbol{x} + a_y\,\boldsymbol{z}\boldsymbol{x} \\ &= (a_x + \boldsymbol{z}\,a_y)\boldsymbol{x}. \end{aligned}$$

Since $\boldsymbol{x}$ is a part of every vector expression of this form, it may be omitted. The remaining form, $a_x + \boldsymbol{z}\,a_y$, is a complex number. Since $\boldsymbol{z}^2$ as a rotor is equivalent to -1, we may write this tensor-rotor in the common form $a + ib$ (where $i^2 = -1$), whose properties are well known.

Again, any vector in the X Y plane may be expressed as a $\boldsymbol{z}$-product, thus,

$$\begin{aligned} \mathsf{A} &= a\,\boldsymbol{z}^n\,\boldsymbol{x} \\ &= a\,(cos\ \theta + \boldsymbol{z}\ sin\ \theta)\ \boldsymbol{x} \\ &= a\ \epsilon^{\boldsymbol{z}\theta}\ \boldsymbol{x}. \end{aligned}$$

Omitting $\boldsymbol{x}$ as before we have left the other two forms of the complex number.

Vector multiplication in the X Y plane with any other rotor than $\boldsymbol{z}$ gives in general imaginary products, i.e., products lying outside of that plane.

FOUR-SPACE ALGEBRA

35. In four-space there are, by definition, four mutually perpendicular axes, X, Y, Z, U. These are so selected that they multiply in circular order, as in 3-space. Each vector is now fully defined by four components. Vectors are added and subtracted as in 3-space.

As in §3 it may be shown that

$$S_{ab} \equiv a_x b_x + a_y b_y + a_z b_z + a_u b_u = a\,b\ cos\ \mathsf{A}\,\mathsf{B},$$

where S_{ab} is, as before, the sum of the $a\,b$ products.

Evidently also when $S_{ab} = 0$, $\mathsf{A} \perp \mathsf{B}$.

36. MULTIPLICATION in 4-space is defined as rotation about the plane* of the multiplying vectors, thru a right angle in the positive direction. The planes of rotation are wholly† perpendicular to the axial plane.

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*Rotation is essentially plane motion. In a 2-flat the axis of rotation is a point. In a 3-flat the axis is a line. In a 4-flat the axis is a plane.

†It is evident from §35, that in 4-space absolutely perpendicular planes exist. For $\mathsf{A} = a_x\,\boldsymbol{x} + a_y\,\boldsymbol{y}$ is any vector in the X Y plane, and $\mathsf{B} = b_z\boldsymbol{z} + b_u\boldsymbol{u}$ is any vector in the Z U plane. Since $S_{ab} = 0$, $\mathsf{A} \perp \mathsf{B}$. That is to say, every vector in the X Y plane is perpendicular to every vector in the Z U plane.

Multiplication of $\mathbf{C}$ by $\mathbf{AB}$, is written $\overline{\mathbf{AB}}\,\mathbf{C}$, and is defined as

(1) Rotation of $\mathbf{C}$ thru 90° in the positive direction about the plane $\mathbf{A\,B}$, and

(2) Simultaneous extension to the length $a\,b\,c$.

By definition, $\overline{\boldsymbol{xy}}\boldsymbol{z} = \boldsymbol{u}$, $\overline{\boldsymbol{yz}}\boldsymbol{u} = \boldsymbol{x}$, $\overline{\boldsymbol{zu}}\boldsymbol{x} = \boldsymbol{y}$, $\overline{\boldsymbol{ux}}\boldsymbol{y} = \boldsymbol{z}$.

Remembering that the plane of rotation is perpendicular to the axial plane it becomes evident that

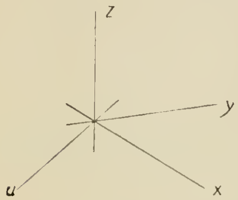

Fig. 9

$$\overline{\boldsymbol{xy}}\boldsymbol{u} = \bar{\boldsymbol{z}},\quad \overline{\boldsymbol{yz}}\boldsymbol{x} = \bar{\boldsymbol{u}},\quad \overline{\boldsymbol{zu}}\boldsymbol{y} = \bar{\boldsymbol{x}},\quad \overline{\boldsymbol{ux}}\boldsymbol{z} = \bar{\boldsymbol{y}}$$

$$\overline{\boldsymbol{xy}}\bar{\boldsymbol{z}} = \bar{\boldsymbol{u}},\quad \overline{\boldsymbol{yz}}\bar{\boldsymbol{u}} = \bar{\boldsymbol{x}},\quad \overline{\boldsymbol{zu}}\bar{\boldsymbol{x}} = \bar{\boldsymbol{y}},\quad \overline{\boldsymbol{ux}}\bar{\boldsymbol{y}} = \boldsymbol{z}$$

$$\overline{\boldsymbol{xy}}\bar{\boldsymbol{u}} = \boldsymbol{z},\quad \overline{\boldsymbol{yz}}\bar{\boldsymbol{x}} = \boldsymbol{u},\quad \overline{\boldsymbol{zu}}\bar{\boldsymbol{y}} = \boldsymbol{x},\quad \overline{\boldsymbol{ux}}\bar{\boldsymbol{z}} = \boldsymbol{y}$$

Coplanar vectors are unchanged in position by 4-space multiplication, because the whole axial plane is unmoved,

$$\overline{\boldsymbol{xy}}\boldsymbol{x} = \boldsymbol{x}$$

$$\overline{\boldsymbol{xy}}\,(a_x\boldsymbol{x} + a_y\boldsymbol{y}) = a_x\boldsymbol{x} + a_y\boldsymbol{y}.$$

37. MULTIPLICATION BY PERPENDICULAR VECTORS.

Let $\mathbf{A} \perp \mathbf{B} \perp \mathbf{C}$, and let $\overline{\mathbf{AB}}\,\mathbf{C} = \mathbf{W}$.

Then $S_{aw} = S_{bw} = S_{cw} = 0$

and $w^2 = w_x^2 + w_y^2 + w_z^2 + w_u^2 = a^2 b^2 c^2$.

Solving for w_x, w_y, w_z, w_u, and collecting,

$$\mathbf{W}_{abc} \equiv \begin{vmatrix} a_x & a_y & a_z & a_u \\ b_x & b_y & b_z & b_u \\ c_x & c_y & c_z & c_u \\ \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{u} \end{vmatrix} \equiv |\, a_x\, b_y\, c_z\, \boldsymbol{u} \,|\,.$$

Also

$$w^2 \equiv |\, a_y\, b_z\, c_u \,|^2 + |\, a_x\, b_z\, c_u \,|^2 + |\, a_x\, b_y\, c_u \,|^2 + |\, a_x\, b_y\, c_z \,|^2$$

$$\equiv \begin{vmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ba} & S_{bb} & S_{bc} \\ S_{ca} & S_{cb} & S_{cc} \end{vmatrix} = a^2\, b^2\, c^2.$$

38. $\mathbf{C}$ is coplanar with $\mathbf{A}, \mathbf{B}$, when

$$\mathbf{C} = m\,\mathbf{A} + n\,\mathbf{B}.$$

Writing the four equations of coordinates, and eliminating m and n, we get the coplanar equations

$$| a_x\, b_y\, c_z | = 0$$

$$| a_y\, b_z\, c_u | = 0.$$

39. To find the perpendicular $\mathbf{N}_2$ from the point $\mathbf{B}$ to the vector $\mathbf{A}$.

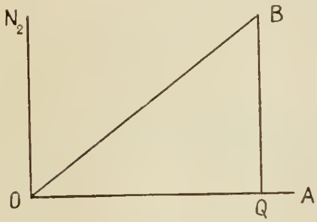

FIG. 10

Let $Q\,B$ be the positive direction of $\mathbf{N}_2$.

Then $q = O\,Q = b \cos \mathbf{A}\,\mathbf{B} = \dfrac{S_{ab}}{a}$.

$$\therefore \quad \mathbf{Q} = \frac{S_{ab}}{a_2}\,\mathbf{A}.$$

$$\therefore \quad \mathbf{N}_2 = \mathbf{B} - \mathbf{Q} = \begin{vmatrix} S_{aa} & S_{ab} \\ \mathbf{A} & \mathbf{B} \end{vmatrix} \div a^2.$$

$$n^2 = b^2 - q^2 = \begin{vmatrix} S_{aa} & S_{ab} \\ S_{ba} & S_{bb} \end{vmatrix} \div a^2 = \frac{v^2}{a^2}.$$

These forms of $\mathbf{N}_2$, n^2, $\mathbf{Q}$, q, are identical in space of four, three and two dimensions, and evidently for space of all dimensions.

In a 2-flat

$$n = \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \div a.$$

40. To find the perpendicular $\mathbf{N}_3$ from $\mathbf{C}$ to the $\mathbf{A}\,\mathbf{B}$ plane.

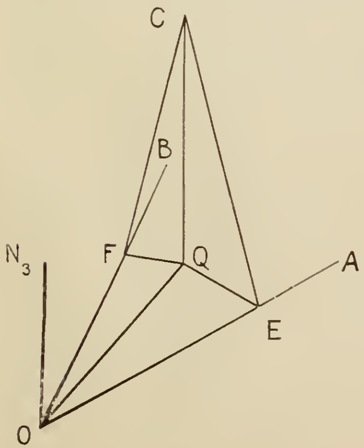

FIG. 11

Let $Q\,C$ be the positive direction of $\mathbf{N}_3$. Draw $Q\,E \perp \mathbf{A}$, $Q\,F \perp \mathbf{B}$, join $C\,F$, $C\,E$.

Then $\quad OE = \dfrac{S_{aq}}{a} = \dfrac{S_{ac}}{a}$

$\therefore \quad S_{aq} = S_{ac}$ .... (1)

and $\quad O\,F = \dfrac{S_{bq}}{b} = \dfrac{S_{bc}}{b}$.

$\therefore \quad S_{bq} = S_{bc}$ .... (2)

Since $\mathbf{A}, \mathbf{B}, \mathbf{Q}_3$, are coplanar

$| a_x\, b_y\, q_z | = 0$ .... (3)

$| a_y\, b_z\, q_u | = 0$ .... (4).

Solving for the coordinates of $\mathbf{Q}_3$, and collecting terms,

$$\mathbf{Q}_3 = (b^2\,\mathrm{S}_{ac} - \mathrm{S}_{ab}\,\mathrm{S}_{bc})\,\frac{\mathbf{A}}{v^2} + (a^2\,\mathrm{S}_{bc} - \mathrm{S}_{ab}\,\mathrm{S}_{ac})\,\frac{\mathbf{B}}{v^2},$$

$$q_3^2 = q_x^2 + q_y^2 + q_z^2 + q_u^2$$

$$= (a^2\,\mathrm{S}_{bc}^2 + b^2\,\mathrm{S}_{ac}^2 - 2\,\mathrm{S}_{ab}\,\mathrm{S}_{ac}\,\mathrm{S}_{bc})\,\frac{1}{v^2},$$

where $$v^2 = a^2 b^2 - \mathrm{S}_{ab}^2\,.$$

Then

$$\mathbf{N}_3 = \mathbf{C} - \mathbf{Q}_3 = \begin{vmatrix} \mathrm{S}_{aa} & \mathrm{S}_{ab} & \mathrm{S}_{ac} \\ \mathrm{S}_{ba} & \mathrm{S}_{bb} & \mathrm{S}_{bc} \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{vmatrix} \div \begin{vmatrix} \mathrm{S}_{aa} & \mathrm{S}_{ab} \\ \mathrm{S}_{ba} & \mathrm{S}_{bb} \end{vmatrix}$$

$$n_3^2 = c^2 - q_3^2 = \begin{vmatrix} \mathrm{S}_{aa} & \mathrm{S}_{ab} & \mathrm{S}_{ac} \\ \mathrm{S}_{ba} & \mathrm{S}_{bb} & \mathrm{S}_{bc} \\ \mathrm{S}_{ca} & \mathrm{S}_{cb} & \mathrm{S}_{cc} \end{vmatrix} \div \begin{vmatrix} \mathrm{S}_{aa} & \mathrm{S}_{ab} \\ \mathrm{S}_{ba} & \mathrm{S}_{bb} \end{vmatrix}$$

$$= \frac{w^2}{v^2}\,.$$

These forms are identical for 3-space, and apparently for all space above it.

In 3-space also $$n_3 = \frac{|\,a_x\, b_y\, c_z\,|}{v}\,.$$

41. To find the normal $\mathbf{N}_4$, from $\mathbf{D}$ to the 3-flat of $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

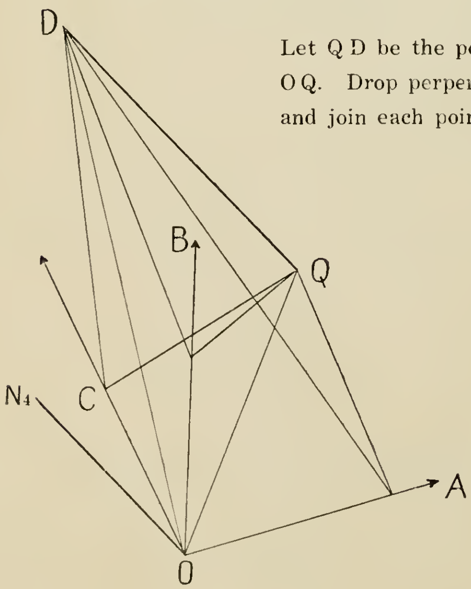

Fig. 12

Let Q D be the positive direction of $\mathbf{N}_4$. Join O Q. Drop perpendiculars from $\mathbf{Q}_4$ on $\mathbf{A}, \mathbf{B}, \mathbf{C}$, and join each point of intersection with $\mathbf{D}$.

Then it is evident as in §40 that

$$\mathrm{S}_{aq} = \mathrm{S}_{ad} \quad \ldots\ldots (1)$$
$$\mathrm{S}_{bq} = \mathrm{S}_{bd} \quad \ldots\ldots (2)$$
$$\mathrm{S}_{cq} = \mathrm{S}_{cd} \quad \ldots\ldots (3).$$

Since $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}_4$ are all in the same 3-flat and therefore all perpendicular to $\mathbf{N}_4$

$$\mathrm{S}_{an} = \mathrm{S}_{bn} = \mathrm{S}_{cn} = \mathrm{S}_{qn} = 0.$$

Eliminating the n's we get the cosolid equation

$$|\,a_x\, b_y\, c_z\, q_u\,| = 0 \ldots (4).$$

Solving for q_x etc. and collecting terms

$$\begin{aligned}\mathbf{Q}_4 = \{ & S_{ad}\,[\mathbf{A}(S_{bc}^2 - b^2c^2) + \mathbf{B}(c^2 S_{ab} - S_{ca} S_{bc}) + \mathbf{C}(b^2 S_{ac} - S_{ab} S_{bc})] \\ & + S_{bd}\,[\mathbf{B}(S_{ac}^2 - a^2c^2) + \mathbf{C}(a^2 S_{bc} - S_{ab} S_{ac}) + \mathbf{A}(c^2 S_{ac} - S_{bc} S_{ac})] \\ & + S_{cd}\,[\mathbf{C}(S_{ab}^2 - a^2b^2) + \mathbf{A}(b^2 S_{ac} - S_{bc} S_{ab}) + \mathbf{B}(a^2 S_{bc} - S_{ac} S_{ab})] \} \div w^2\end{aligned}$$

Therefore

$$\mathbf{N}_4 = \mathbf{D} - \mathbf{Q}_4 = \begin{vmatrix} S_{aa} & S_{ab} & S_{ac} & S_{ad} \\ S_{ba} & S_{bb} & S_{bc} & S_{bd} \\ S_{ca} & S_{cb} & S_{cc} & S_{cd} \\ \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{vmatrix} \div \begin{vmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ba} & S_{bb} & S_{bc} \\ S_{ca} & S_{cb} & S_{cc} \end{vmatrix}.$$

$$n_4^2 = \frac{|\, S_{aa}\; S_{bb}\; S_{cc}\; S_{dd} \,|}{|\, S_{aa}\; S_{bb}\; S_{cc} \,|} = \frac{|\, a_x\; b_y\; c_z\; d_u \,|^2}{w^2}.$$

42. RELATION OF $\mathbf{N}_4$ TO THE RECTOR $\mathbf{W}$. Since $\mathbf{N}_4$ and $\mathbf{W}$ are each perpendicular to the 3-flat of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, they differ only in their tensors,

Hence $\qquad \mathbf{N}_4 = \dfrac{n_4}{w}\mathbf{W} = \dfrac{|\, a_x\; b_y\; c_z\; d_u \,|}{w^2}\mathbf{W}.$

Similarly, in 3-space

$$\mathbf{N}_3 = \frac{n_3}{v}\mathbf{V} = \frac{|\, a_x\; b_y\; c_z \,|}{v^2}\mathbf{V}.$$

And in 2-space

$$\mathbf{N}_2 = \frac{n_2}{f}\mathbf{F} = \frac{|\, a_x\; b_y \,|}{f^2}\mathbf{F},$$

when $\qquad \mathbf{F} = \begin{vmatrix} a_x & a_y \\ x & y \end{vmatrix}$ is the perpendicular to $\mathbf{A}$.

The forms for $\mathbf{N}_3$, $\mathbf{N}_2$, n_3^2, n_2^2, may be obtained by suppressing rows and columns in the determinant forms of $\mathbf{N}_4$, n_4^2. It is evident that we have here a correspondence between the geometric space-form for a perpendicular and the algebraic space-form or matrix, which is true for all space.

43. To find the product $\overline{\mathbf{AB}}^{\,n}\,\mathbf{C}$ when n is real.

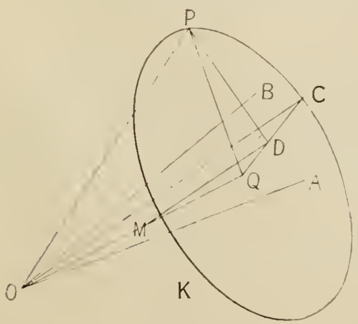

FIG. 13

Let CPK be the circle of rotation of the point $\mathbf{C}$, and let Q be its centre in the AB plane.

Join QO, QC, QP.

Let OP be the position of OC after rotation, so that $OP = \overline{\boldsymbol{ab}}^{\,n}\,\mathbf{C}$.

Draw PD $\perp$ QC, DM $\parallel$ CO.

The angle $CQP = \theta = n\dfrac{\pi}{2}$.

Then since P D, being in the plane of rotation, is perpendicular to the $\mathbf{A}\,\mathbf{B}$ plane, and also to Q C; P D is perpendicular to the 3-flat of A, B, C, and is therefore parallel to $\mathbf{W}$.

$$\mathrm{OM} = \mathrm{OQ}\,\frac{\mathrm{CD}}{\mathrm{CQ}} = q\ vers\ \theta$$

$$\mathrm{MD} = \mathrm{OC}\,\frac{\mathrm{QD}}{\mathrm{QC}} = c\ cos\ \theta$$

$$\mathrm{DP} = \mathrm{PQ}\ sin\ \theta = n_3\ sin\ \theta = \frac{w}{v}\ sin\ \theta.$$

As vectors

$$\mathrm{OP} = \mathrm{OM} + \mathrm{MD} + \mathrm{DP},$$

$$\therefore\quad \overline{\boldsymbol{a}\boldsymbol{b}}^{n}\,\mathbf{C} = \mathbf{Q}_3\ vers\ \theta + \mathbf{C}\ cos\ \theta + \frac{\mathbf{W}}{v}\ sin\ \theta$$

$$\therefore\quad \overline{\mathbf{A}\mathbf{B}}^{n}\,\mathbf{C} = a^n\,b^n\left[\left\{\mathbf{A}\,(b^2\,S_{ac} - S_{ab}\,S_{bc}) + \mathbf{B}(a^2\,S_{bc} - S_{ab}\,S_{ac})\right\}\frac{vers\ \theta}{v^2} + \mathbf{C}\ cos\ \theta + \frac{\mathbf{W}}{v}\ sin\ \theta\right].$$

44. If $\mathbf{C}$ is perpendicular to $\mathbf{A}$ and $\mathbf{B}$, then

$$\overline{\mathbf{A}\mathbf{B}}^{n}\,\mathbf{C} = a^n\,b^n\left\{\mathbf{C}\ cos\ \theta + \frac{\mathbf{W}}{v}\ sin\ \theta\right\},$$

and

$$\overline{\boldsymbol{a}\boldsymbol{b}}\,\mathbf{C} = \frac{\mathbf{W}}{v}.$$

$$\therefore\quad \overline{\mathbf{A}\mathbf{B}}^{n}\,\mathbf{C} = a^n\,b^n\,(cos\ \theta + \overline{\boldsymbol{a}\boldsymbol{b}}\ sin\ \theta)\,\mathbf{C}$$

$$= a^n\,b^n\,\epsilon^{\overline{\boldsymbol{a}\boldsymbol{b}}\theta}\,\mathbf{C}.$$

The rotor $e^{\overline{\boldsymbol{a}\boldsymbol{b}}\theta}$ resembles the rotor $e^{\boldsymbol{a}\theta}$ found in 3-space multiplication. It is evident that similar rotors (quaternions) will be found in all higher space forms.

45. The intersections of loci are found as in §32.

Example 1. Find the intersection of the 3-flat

$$lll\,(\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{y} + \boldsymbol{c}\boldsymbol{z})$$

with the helix

$$l\left\{\overline{\boldsymbol{x}\boldsymbol{y}}^{n}(\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}) + n\boldsymbol{x}\right\} = l(\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}\ cos\ \theta + \boldsymbol{u}\ sin\ \theta + n\boldsymbol{x}).$$

Equating coefficients of $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{u}$,

$$a = n + 1$$
$$b = 1$$
$$c = cos\ \theta$$
$$0 = sin\ \theta.$$
$$\therefore\quad c = \pm 1$$
and $n = 0, \pm 2, \pm 4$, etc.

The intersection is

$$\ell^{n} (n+1)\,\mathbf{x} + \mathbf{y} \pm \mathbf{z},$$

representing two rows of points parallel to X.

EXAMPLE 2.—Find the intersection of the plane $\ell\ell\,(a\mathbf{x} + b\mathbf{y})$ with the solid cylinder

$$\ell_0^{4}\,\ell_0^{c}\,\ell_0^{m} \left\{ \overline{\mathbf{x}\mathbf{y}}^{\,n}\,\mathbf{C} + m\,(\mathbf{x} + \mathbf{u}) \right\} \equiv \ell\ell\ell \left[c_x\mathbf{x} + c_y\mathbf{y} + (c_z \cos\theta - c_u \sin\theta)\mathbf{z} + (c_u \cos\theta + c_z \sin\theta)\,\mathbf{u} \right].$$

Equating coefficients of $\mathbf{x}$, $\mathbf{y}$,

$$a = c_x + m$$
$$b = c_y,$$

and the plane locus becomes the rectangle

$$\ell_0^{c}\,\ell_0^{m} \left\{ (c_x + m)\,\mathbf{x} + c_y\mathbf{y} \right\}.$$

46. PROJECTIONS. To express any vector $\mathbf{K}$ in terms of any four vectors $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, not in one 3-flat, write

$$l\mathbf{A} + m\mathbf{B} + n\mathbf{C} + r\mathbf{D} = \mathbf{K}.$$

Then
$$l a_x + m b_x + n c_x + r d_x = k_x$$
$$l a_y + m b_y + n c_y + r d_y = k_y$$
$$l a_z + m b_z + n c_z + r d_z = k_z$$
$$l a_u + m b_u + n c_u + r d_u = k_u$$

$$\therefore\quad l = \frac{|\,k_x\; b_y\; c_z\; d_u\,|}{|\,a_x\; b_y\; c_z\; d_u\,|}, \qquad m = \frac{|\,a_x\; k_y\; c_z\; d_u\,|}{|\,a_x\; b_y\; c_z\; d_u\,|},$$

$$n = \frac{|\,a_x\; b_y\; k_z\; d_u\,|}{|\,a_x\; b_y\; c_z\; d_u\,|}, \qquad r = \frac{|\,a_x\; b_y\; c_z\; k_u\,|}{|\,a_x\; b_y\; c_z\; d_u\,|}.$$

Writing either l, m, n or r equal to zero the remaining terms of $\mathbf{K}$ are the projection of $\mathbf{K}$ made parallel to the vanishing vector and upon the 3-flat of the remaining vectors. To project $\mathbf{K}$ normally upon the 3-flat of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, write $\mathbf{D} = \mathbf{W}_{abc}$, then make $r = 0$.

The sum of any two terms of $\mathbf{K}$ is the projection of $\mathbf{K}$ upon their plane, made parallel to the plane of the other two vectors.

Loci are projected in the same way.

47. As an illustration of the method of the last section we may find the principal orthogonal projections* of the regular 8-cubed tessaract whose edge is unity,

$$\int_0^1\int_0^1\int_0^1\int_0^1 \mathsf{K}, \text{ upon the 3-flats about it.}$$

(1) Parallel to $\boldsymbol{x}$, on the 3-flat of $\boldsymbol{y}$, $\boldsymbol{z}$, $\boldsymbol{u}$, the projection is obtained by writing $k_x = 0$, giving the cube

$$\int_0^1\int_0^1\int_0^1 (k_y\,\boldsymbol{y} + k_z\,\boldsymbol{z} + k_u\,\boldsymbol{u}).$$

(2) Parallel to $\boldsymbol{x} + \boldsymbol{y}$. Let $\boldsymbol{x} + \boldsymbol{y} = \mathsf{D}$.

To get three other rectangular vectors we may take

$$\mathsf{A} = \boldsymbol{z}$$

$$\mathsf{B} = \boldsymbol{u}$$

$$\mathsf{C} = \mathsf{W}_{abd} = \boldsymbol{y} - \boldsymbol{x}.$$

Then $l = k_z$, $m = k_u$, $n = \dfrac{k_y - k_x}{2}$.

Writing $r = 0$ the projection becomes

$$\int_0^1\int_0^1\int_0^1\int_0^1 \left\{ k_z\,\mathsf{A} + k_u\,\mathsf{B} + \frac{k_y - k_x}{2}\,\mathsf{C} \right\}.$$

And $a = 1$, $b = 1$, $c = \sqrt{2}$.

To express this locus in geometrical terms we note first that since it contains three vectors, not coplanar, with independent variable coefficients, it is a 3-space solid; and in the second place that the original axes, $\boldsymbol{z}$, $\boldsymbol{u}$, which are perpendicular to the line of projection, remain unchanged. The axes $\boldsymbol{x}$, $\boldsymbol{y}$, are each foreshortened in the ratio of $\sqrt{2} : 1$. Projecting $\boldsymbol{x}$ and $\boldsymbol{y}$ by the same plan as for K we get for the projections

$$\boldsymbol{x}^1 = \tfrac{1}{2}\,\mathsf{C}$$

$$\boldsymbol{y}^1 = -\tfrac{1}{2}\,\mathsf{C}$$

making the total distance $\sqrt{2}$ along C.

Consider next the variables in the locus. k_z and k_u are entirely independent, with limits from 0 to 1, and $\int_0^1\int_0^1 (k_z\,\mathsf{A} + k_u\,\mathsf{B})$ is a square in the $\mathsf{A}\,\mathsf{B}$ plane. The solid is a right square prism whose extension along C is given by the last term $\dfrac{k_y - k_x}{2}\mathsf{C}$ of the locus. k_y and k_x vary independently from

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*For a purely geometric investigation of these projections see the American Journal of Mathematics, Volume XV, No. 2, pages 179-189.

0 to 1. The lower limit of the term occurs when $k_y = 0,\ k_x = 1$, namely, $-\frac{1}{2}\,\mathsf{C}$; and the upper limit is $\frac{1}{2}\,\mathsf{C}$. Since $c = \sqrt{2}$, the length of the prism is $\sqrt{2}$ along C.

(3) Parallel to $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}$, $(= \mathsf{D})$.

Take for the other rectangular axes

$$\mathsf{A} = \boldsymbol{u}$$

$$\mathsf{B} = \boldsymbol{x} - \boldsymbol{y}$$

$$\mathsf{C} = \mathrm{W}_{abd} = \boldsymbol{x} + \boldsymbol{y} - 2\,\boldsymbol{z}.$$

Then $l = k_u,\ m = \dfrac{k_x - k_y}{2},\ \ n = \dfrac{k_x + k_y - 2\,k_z}{6}$.

Put $r = 0$; the projection is

$$\int_0^1\int_0^1\int_0^1\int_0^1 \left\{ k_u\,\mathsf{A} + \frac{k_x - k_y}{2}\,\mathsf{B} + \frac{k_x + k_y - 2\,k_z}{6}\,\mathsf{C} \right\},$$

where $a = 1,\ \ b = \sqrt{2},\ c = \sqrt{6}$.

The figure is again a 3-solid; the axis $\boldsymbol{u}$, perpendicular to D, remaining unchanged. Projecting the other three axes we get

$$\boldsymbol{x}^1 = \tfrac{1}{2}\,\mathsf{B} + \tfrac{1}{6}\,\mathsf{C}$$

$$\boldsymbol{y}^1 = -\tfrac{1}{2}\,\mathsf{B} + \tfrac{1}{6}\,\mathsf{C}$$

$$\boldsymbol{z}^1 = -\tfrac{1}{3}\,\mathsf{C}.$$

The length of each of these is $\frac{\sqrt{6}}{3}$. This length may be found directly by the equation

$$\sin^2 \boldsymbol{x}\mathsf{D} = 1 - \cos^2 \boldsymbol{x}\mathsf{D} = 1 - \frac{\mathrm{S}^2_{xd}}{d^2} = \tfrac{2}{3}.$$

The variable k_u is independent. The figure is therefore a right prism of unit length along A. To find the prism base, or section in the $\mathsf{B}\,\mathsf{C}$ plane, draw the axes $\frac{1}{2}\,\mathsf{B}$, $\frac{1}{3}\,\mathsf{C}$, and plot the figure.

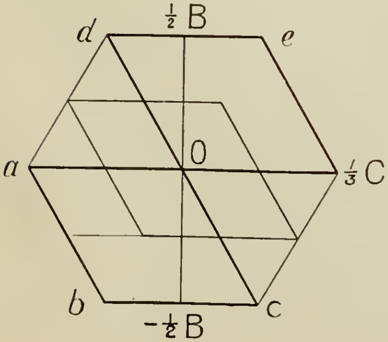

FIG. 14

First, let $k_x = k_y = 0$, while k_z varies from 0 to 1, tracing the line along C from $-\frac{1}{3}$ to 0, the line $a\,0$. Next let $k_y = 1$; the locus of k_z is then the line bc from

$$-\frac{\mathsf{B}}{2} - \frac{\mathsf{C}}{6} \text{ to } -\frac{\mathsf{B}}{2} + \frac{\mathsf{C}}{6}.$$

Intermediate values of k_y fill out the parallelogram $a\,c$.

Next let $k_x = 1$, and proceed as before, obtaining the parallelogram dC, whose limiting lines are de from $\frac{\mathsf{B}}{2} - \frac{\mathsf{C}}{6}$ to $\frac{\mathsf{B}}{2} + \frac{\mathsf{C}}{6}$, and OC from O to $\frac{\mathsf{C}}{3}$.

Intermediate values of k_x give similar parallelograms commencing at every point along ad and covering the regular hexagon ace. The whole projection is a right hexagonal prism. The projected axes $\boldsymbol{x}^1$, $\boldsymbol{y}^1$, $\boldsymbol{z}^1$, are Oe, Oc, Oa.

(4) Parallel to $\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z} + \boldsymbol{u}$, $(= \mathsf{D})$.

Take for the other rectangular axes

$$\mathsf{A} = \boldsymbol{x} - \boldsymbol{y}$$

$$\mathsf{B} = \boldsymbol{z} - \boldsymbol{u}$$

$$\mathsf{C} = \tfrac{1}{2}\,\mathsf{W}_{abd} = \boldsymbol{x} + \boldsymbol{y} - \boldsymbol{z} - \boldsymbol{u}.$$

Then $l = \frac{k_x - k_y}{2}$, $m = \frac{k_z - k_u}{2}$, $n = \frac{k_x + k_y - k_z - k_u}{4}$,

and the locus of the projection is

$$\int_0^1\int_0^1\int_0^1\int_0^1 \left\{ \frac{k_x - k_y}{2}\mathsf{A} + \frac{k_z - k_u}{2}\mathsf{B} + \frac{k_x + k_y - k_z - k_u}{4}\mathsf{C} \right\},$$

where $a = b = \sqrt{2}$, $c = 2$.

Projecting the axes $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{u}$, we get

$$\boldsymbol{x}^1 = \tfrac{1}{2}\mathsf{A} + \tfrac{1}{4}\mathsf{C}$$

$$\boldsymbol{y}^1 = -\tfrac{1}{2}\mathsf{A} + \tfrac{1}{4}\mathsf{C}$$

$$\boldsymbol{z}^1 = \tfrac{1}{2}\mathsf{B} - \tfrac{1}{4}\mathsf{C}$$

$$\boldsymbol{u}^1 = -\tfrac{1}{2}\mathsf{B} - \tfrac{1}{4}\mathsf{C}$$

and the length of each projected axis is $\frac{\sqrt{3}}{2}$.

To obtain the geometric form of the projection, give to all the variables the value zero, then to each one separately give all values up to unity. This gives four lines from O, identical with the projected axes. With three of these lines as adjacent edges form a parallelopiped, and form three more parallelopipeds with the three other possible groups of the four lines. The sum of these four solids, a rhombic dodekahedron, is the projection required.